TEMPERATURE DISTRIBUTION IN LAMINAR FLOWS AT LARGE SHEAR STRESSES

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The energy equation is solved for the case of laminar flow of a compressible Newtonian liquid with allowance for heat of friction. Theoretical temperature profiles are presented for flow in channels with constant and linearly varying capillary wall temperatures. The theoretical values of the mean temperature are in good agreement with the experimental results.

When liquids move through capillaries under a pressure of several tens of atmospheres, the thermal effects of energy dissipation and compressibility become appreciable. These phenomena were reviewed in [1], where it was shown that the magnitudes of both effects are of the same order and that, in general, in internal flows the compressibility cannot be neglected.

In the first approximation, in flow through a thermally insulated capillary the average temperature of the liquid increases by about 1° C for each 20–30 atm of pressure drop.

Under these conditions the simple forms of the equations of motion and energy, which hold true for flows in the region of small pressure drops, cease to be valid. Attention was first drawn to this by Hersey [2], who correctly attributed the experimentally observed deviations from the Poiseuille law to the effect of the heat of friction on the temperature field of the liquid and hence on its viscosity.

It is customary to distinguish two types of forced convection: a) adiabatic flow when there is no heat exchange with the ambient medium through the capillary walls; b) flow through an isothermal capillary at whose walls the initial (at the channel inlet) temperature of the liquid is kept constant.

Even recent publications confine themselves, with a few exceptions, to the problem of temperature distribution and give different variants of solutions of the energy equation. Some authors propose a solution of the energy equation that neglects the convection term [1–6]. Omitting the convective component limits the applicability of the solution to the region of the steady-state temperature profile. This region is important in connection with very long isothermal capillaries, in which equilibrium has been established between the dissipative component and the heat transmitted through the walls (in this case the temperature is a function of the radius only [1]).

Solutions for incompressible liquids in insulated or isothermal capillaries with allowance for convection are offered in [7–11].

Kudryashev and Golovin [12] have examined the effect of energy dissipation on heat transfer in a laminar channel flow of incompressible liquid at constant wall temperature.

Problems of the capillary flow of compressible liquids were first considered by Toor [1]. In [13] the same author discussed flow through an isothermal capillary with allowance for convection.

The solution proposed by Madejski [14] for a compressible liquid holds true only for a perfect gas [15]. A general mathematical model of laminar flow with allowance for heat of friction and expansion has been developed by Gee and Lyon [16]. These authors examined the flow of a non-Newtonian liquid in a heated or cooled channel with allowance for the variation of the physical parameters of the liquid. After publication of the work of Gee and Lyon the problem might have been considered solved in principle, if the nonlinear differential equations they obtained had been amenable to analytic solution. The authors used computers to find the temperature and velocity profiles as a function of the radius and length of the channel for several specific cases, obtaining values that agreed with the experimental results.

In this paper the problem is again examined by means of a simplified method based on the independence of the solutions of the equations of motion and energy. This is one of the approaches recommended by certain authors [17]; it may be regarded as the first stage of an iteration method.

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Formulation of the problem. The energy equation for a Newtonian liquid with constant thermal conductivity can be written in one of its alternative forms [18] as

$$\gamma c_p \frac{DT}{Dt} = \lambda \nabla^2 T + T\beta \frac{Dp}{Dt} + \mu\Phi, \qquad (1)$$

where the coefficient of isobaric thermal expansion

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p. \qquad (2)$$

This is the general equation describing the temperature distribution for a real, dissipatively heated liquid in the absence of other internal heat sources (sinks).

Equation (1) has been solved on the basis of the following assumptions:

1. The physical parameters of the liquid are constant and do not depend on temperature and pressure (with the reservation mentioned in item 6).
2. The motion of the liquid has been stabilized.
3. The circumferential and radial components of velocity can be neglected. The velocity distribution in cross sections of the channel is parabolic.

Table 1

Numerical Values of the Characteristic Constants A_i

i	λ_i	A'_i	A''_i	A'''_i
1	7.3153	+1.46622	+0.3360	—0.8020
2	44.6090	—0.802476	—0.1254	+0.0719
3	113.921	+0.587094	+0.0579	—0.0207
4	215.241	—0.474897	—0.0330	+0.0139

4. The effect of external forces is neglected.

5. Heat conduction in the direction of the channel axis may be disregarded.

6. In accordance with [1] in a certain temperature interval it is possible to use the mean value of the product $T\beta = \varepsilon$.

With these assumptions Eq. (1) may be written in the simplified form [1]

$$\gamma c_p V \frac{\partial T}{\partial x} = \lambda \frac{1}{\gamma} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + $$
$$+ \varepsilon V \frac{\partial p}{\partial x} + \mu \left(\frac{\partial V}{\partial r} \right)^2. \qquad (3)$$

In Eq. (3) we have taken into account the appreciable influence of the compressibility of the liquid on the temperature distribution and have omitted the less important effect on the velocity and dissipation.

Under these conditions the equation of motion may be written in the form

$$\frac{dp}{dx} = \frac{1}{r} \frac{d}{dr}\left(\mu r \frac{dV}{dr} \right). \qquad (4)$$

With the above assumptions Eqs. (3) and (4) are independent. Replacing the pressure gradient with the average pressure drop along the capillary $dp/dx = (-P/x)$, on the basis of Eqs. (3) and (4) we can write

$$\gamma c_p \left(\frac{P}{x} \right) \frac{R^2}{4\mu} \left[1 - \left(\frac{r}{R} \right)^2 \right] \frac{\partial T}{\partial x} =$$
$$= \lambda \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + \varepsilon \frac{R^2}{4\mu} \left[1 - \right.$$
$$\left. - \left(\frac{r}{R} \right)^2 \left(\frac{P}{x} \right)^2 \right] + \frac{r^2}{4\mu} \left(\frac{P}{x} \right)^2. \qquad (5)$$

After rearranging and introducing the relative radius $\rho = r/R$, together with Brinkman's relative length [7]

$$z = \frac{4\lambda\mu x}{\gamma c_p R^4 (P/x)} = \frac{1}{\mathrm{Pe}} \frac{x}{R} = \frac{\pi}{2} (\mathrm{Gz})^{-1} \qquad (6)$$

and the relative temperature

$$\tau = \frac{16\mu\lambda T}{R^4 (P/x)^2} = 2\pi (\mathrm{Gz})^{-1} \left(\frac{c_p \gamma T}{P} \right), \qquad (7)$$

we obtain the final form of the differential equation of temperature distribution for laminar flow of a Newtonian compressible liquid with allowance for the heat of friction:

$$(1 - \rho^2) \frac{\partial \tau}{\partial z} =$$
$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \tau}{\partial \rho} \right) + 4 [(1 + \varepsilon) \rho^2 - \varepsilon]. \qquad (8)$$

In this paper Eq. (8) has been solved for a Newtonian compressible liquid flowing in a capillary with a constant arbitrary wall temperature and in a capillary with a wall temperature that varies linearly with length.

Solution of the equations. A. Wall temperature constant. This case is characterized by the following boundary conditions:

$$\tau(\rho, z), \quad \tau(1, z) = \tau_w = \mathrm{const};$$
$$0 \leqslant \rho \leqslant 1, \quad \tau(\rho, 0) = \tau_0;$$
$$0 \leqslant z \leqslant \infty, \quad \left(\frac{\partial \tau}{\partial \rho} \right)_{\rho=0} = 0. \qquad (9)$$

Introducing the new variable

$$\theta = (\tau - \tau_w)/(\tau_0 - \tau_w), \qquad (10)$$

we can write Eqs. (9) in the form

$$(1 - \rho^2) \frac{\partial \theta}{\partial z} =$$
$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \theta}{\partial \rho} \right) + \frac{4(1+\varepsilon)\rho^2}{(\tau_0 - \tau_w)} - \frac{4\varepsilon}{(\tau_0 - \tau_w)}. \qquad (11)$$

In the new notation the boundary conditions are

$$\theta(\rho, z), \quad \theta(1, z) = 0;$$
$$\theta(\rho, 0) = 1;$$
$$\left(\frac{\partial \theta}{\partial \rho} \right) = 0. \qquad (12)$$

We will find [7, 9] the solution of Eq. (12) in the form of a sum of two functions

$$\theta = \theta_1 + \theta_2, \qquad (13)$$

where θ_1 is the solution for the case $(\partial\theta/\partial z) = 0$, and θ_2 is an auxiliary function required to obtain the general solution.

The first term is found on the assumption that the convection term may be neglected, $(\partial\theta/\partial z) = 0$. We then obtain

$$\theta_1 = \frac{1}{4} [(1 + \varepsilon)(1 - \rho^4) - 4\varepsilon (1 - \rho^2)] \frac{1}{(\tau_0 - \tau_w)}. \qquad (14)$$

Substituting (14) in (13) and then in (11), we reduce the problem to the classical case of heat transfer in a laminar flow:

$$(1 - \rho^2) \frac{\partial \theta_2}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \theta_2}{\partial \rho} \right). \qquad (15)$$

Representing the function θ_2 in the form of a product

$$\theta_2(\rho, z) = \Phi(\rho) \psi(z),$$

Table 2

Numerical Values of the Auxiliary Integrals M and N

ε	$\tau_0-\tau_w$	Integrals	z						
			1.0	0.5	0.2	0.1	0 05	0.02	0.01
0	0	*M*	$8.329\cdot 10^{-2}$	8.163	6.805	5.140	3.635	2.163	1.410
		N	$3.124\cdot 10^{-2}$	3.075	2.673	2.178	1.699	1.144	$8.043\cdot 10^{-3}$
0.1	0	*M*	$6.663\cdot 10^{-2}$	6.537	5.501	4.233	3.069	1.885	1.252
		N	$2.603\cdot 10^{-2}$	2.260	1.882	1.882	1.506	1.042	$7.437\cdot 10^{-3}$
0.15	0	*M*	$5.831\cdot 10^{-2}$	5.724	4.851	3.780	2.786	1.746	1.174
		N	$2.343\cdot 10^{-2}$	2.311	2.054	1.734	2.409	$9.906\cdot 10^{-3}$	7.134
0.2	0	*M*	$4.998\cdot 10^{-2}$	4.911	4.200	3.327	2.503	1.607	1.095
		N	$2.083\cdot 10^{-2}$	2.057	1.847	1.586	1.312	$9.396\cdot 10^{-3}$	6.831
0.3	0	*M*	$3.332\cdot 10^{-2}$	3.285	2.899	2.420	1.937	1.329	$9.368\cdot 10^{-3}$
		N	$1.562\cdot 10^{-2}$	1.548	1.434	1.289	1.118	$8.376\cdot 10^{-3}$	6.226
0.4	0	*M*	$1.666\cdot 10^{-2}$	1.659	1.597	1.513	1.372	1.051	$7.790\cdot 10^{-3}$
		N	$1.042\cdot 10^{-2}$	1.039	1.021	$9.931\cdot 10^{-3}$	9.248	7.356	5.620
0.5	0	*M*	$8.501\cdot 10^{-6}$	$3.296\cdot 10^{-4}$	$2.958\cdot 10^{-3}$	6.064	8.059	7.730	6.211
		N	$5.211\cdot 10^{-3}$	5.306	6.081	6.969	7.312	6.335	5.014
0	+0.1	*M*	$8.331\cdot 10^{-2}$	8.237	7.471	6.536	5.708	4.962	4.638
		N	$3.124\cdot 10^{-2}$	3.097	2.870	2.592	2.336	2.078	1.956
0	−0.1	*M*	$8.327\cdot 10^{-2}$	8.088	6.135	3.744	1.561	$-6.363\cdot 10^{-3}$	$-1.817\cdot 10^{-2}$
		N	$3.123\cdot 10^{-2}$	3.053	2.476	1.764	1.062	$2.093\cdot 10^{-3}$	$-3.476\cdot 10^{-3}$
0	+0.2	*M*	$8.333\cdot 10^{-2}$	8.312	8.139	7.931	7.782	7.761	7.865
		N	$3.125\cdot 10^{-2}$	3.119	3.068	3.008	2.973	3.012	3.108
0	−0.2	*M*	$8.325\cdot 10^{-2}$	8.014	5.467	2.349	$-5.127\cdot 10^{-3}$	$-3.435\cdot 10^{-2}$	$-5.045\cdot 10^{-2}$
		N	$3.123\cdot 10^{-2}$	3.031	2.279	1.349	$4.254\cdot 10^{-3}$	$-7.250\cdot 10^{-3}$	$-1.400\cdot 10^{-2}$
0	+0.5	*M*	$8.334\cdot 10^{-2}$	8.535	10.143	12.119	14.003	16.158	17.547
		N	$3.127\cdot 10^{-2}$	3.184	3.659	4.252	4.884	5.815	6.563
0	−0.5	*M*	$8.319\cdot 10^{-2}$	7.791	3.463	$-1.839\cdot 10^{-2}$	$-6.734\cdot 10^{-2}$	$-1.183\cdot 10^{-1}$	$-1.473\cdot 10^{-1}$
		N	$3.121\cdot 10^{-2}$	2.965	1.687	$1.050\cdot 10^{-3}$	$-1.485\cdot 10^{-2}$	$-3.528\cdot 10^{-2}$	$-4.955\cdot 10^{-2}$
0.2	+0.1	*M*	$5.000\cdot 10^{-2}$	4.985	4.868	4.722	4.577	4.406	4.322
		N	$2.083\cdot 10^{-2}$	2.079	2.044	2.000	1.949	1.874	1.835
0.2	−0.1	*M*	$4.996\cdot 10^{-2}$	4.836	3.532	1.931	$4.294\cdot 10^{-3}$	$-1.192\cdot 10^{-2}$	$-2.133\cdot 10^{-2}$
		N	$2.082\cdot 10^{-2}$	2.035	1.650	1.171	$6.752\cdot 10^{-3}$	$5.293\cdot 10^{-5}$	$-4.688\cdot 10^{-3}$
0.2	+0.2	*M*	$5.001\cdot 10^{-2}$	5.060	5.536	6.118	6.650	7.205	7.549
		N	$2.084\cdot 10^{-2}$	2.101	2.241	2.415	2.586	2.808	2.987
0.2	−0.2	*M*	$4.994\cdot 10^{-2}$	4.762	2.864	$5.352\cdot 10^{-3}$	$-1.644\cdot 10^{-2}$	$-3.991\cdot 10^{-2}$	$+5.360\cdot 10^{-2}$
		N	$2.082\cdot 10^{-2}$	2.013	1.453	$7.564\cdot 10^{-3}$	$3.819\cdot 10^{-4}$	$-9.290\cdot 10^{-3}$	$-1.621\cdot 10^{-2}$
0.2	+0.5	*M*	$5.007\cdot 10^{-2}$	5.283	7.540	10.305	12.871	15.602	17.231
		N	$2.085\cdot 10^{-2}$	2.167	2.833	3.659	4.500	5.611	6.442
0.2	−0.5	*M*	$4.988\cdot 10^{-2}$	4.539	$8.599\cdot 10^{-3}$	$-3.652\cdot 10^{-2}$	$-7.865\cdot 10^{-2}$	$-1.239\cdot 10^{-1}$	$-1.504\cdot 10^{-1}$
		N	$2.080\cdot 10^{-2}$	1.947	$8.610\cdot 10^{-3}$	$-4.875\cdot 10^{-3}$	$-1.875\cdot 10^{-2}$	$-3.732\cdot 10^{-2}$	$-5.076\cdot 10^{-2}$
0.4	+0.1	*M*	$1.668\cdot 10^{-2}$	1.733	2.265	2.909	3.445	3.850	4.006
		N	$1.042\cdot 10^{-2}$	1.061	1.218	1.408	1.562	1.670	1.714
0.4	−0.1	*M*	$1.665\cdot 10^{-2}$	1.585	$9.292\cdot 10^{-3}$	$1.174\cdot 10^{-3}$	$-7.020\cdot 10^{-3}$	$-1.748\cdot 10^{-3}$	$-2.448\cdot 10^{-2}$
		N	$1.041\cdot 10^{-2}$	1.017	$8.239\cdot 10^{-3}$	$5.785\cdot 10^{-3}$	$2.879\cdot 10^{-3}$	$-1.988\cdot 10^{-3}$	$-5.899\cdot 10^{-2}$
0.4	+0.2	*M*	$1.670\cdot 10^{-2}$	1.808	2.935	4.305	5.519	6.649	7.233
		N	$1.043\cdot 10^{-2}$	1.083	1.416	1.822	2.199	2.604	2.866
0.4	−0.2	*M*	$1.663\cdot 10^{-2}$	1.510	$2.612\cdot 10^{-3}$	$1.278\cdot 10^{-2}$	$-2.776\cdot 10^{-2}$	$-4.547\cdot 10^{-2}$	$-5.676\cdot 10^{-3}$
		N	$1.040\cdot 10^{-2}$	$9.955\cdot 10^{-3}$	$6.267\cdot 10^{-3}$	$1.639\cdot 10^{-3}$	$-3.490\cdot 10^{-3}$	$-1.133\cdot 10^{-2}$	$-1.742\cdot 10^{-2}$
0.4	+0.5	*M*	$1.676\cdot 10^{-2}$	2.031	4.937	8.492	11.740	15.050	16.916
		N	$1.044\cdot 10^{-2}$	1.149	2.007	3.066	4.110	5.407	6.321
0.4	−0.5	*M*	$1.657\cdot 10^{-2}$	$1.287\cdot 10^{-2}$	$-1.743\cdot 10^{-2}$	$-5.466\cdot 10^{-2}$	$-8.997\cdot 10^{-2}$	$-1.294\cdot 10^{-1}$	$-1.536\cdot 10^{-1}$
		N	$1.039\cdot 10^{-2}$	$9.296\cdot 10^{-3}$	$3.500\cdot 10^{-4}$	$-1.080\cdot 10^{-2}$	$-2.260\cdot 10^{-2}$	$-3.936\cdot 10^{-2}$	$-5.197\cdot 10^{-2}$

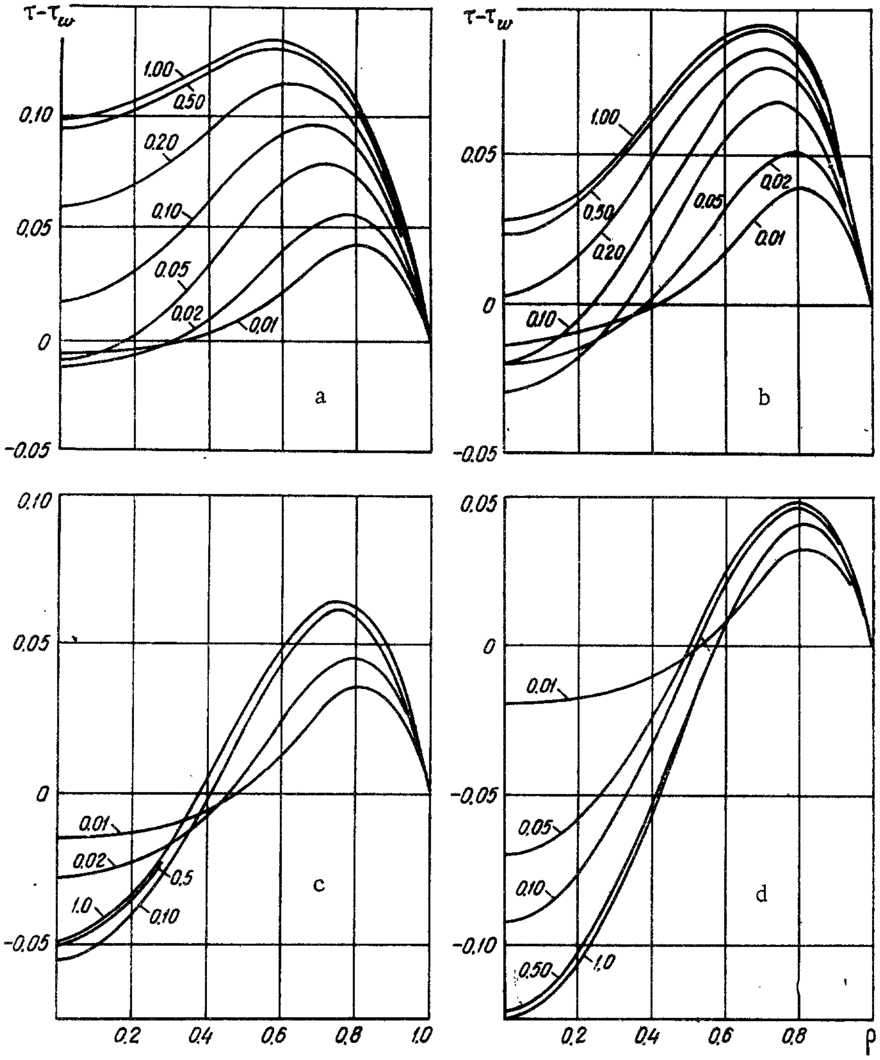

Fig. 1. Temperature distribution at $\tau_W = \tau_0$: a) at $\varepsilon = 0.2$; b) 0.3; c) 0.4; d) 0.5. The figures on the curves are values of z.

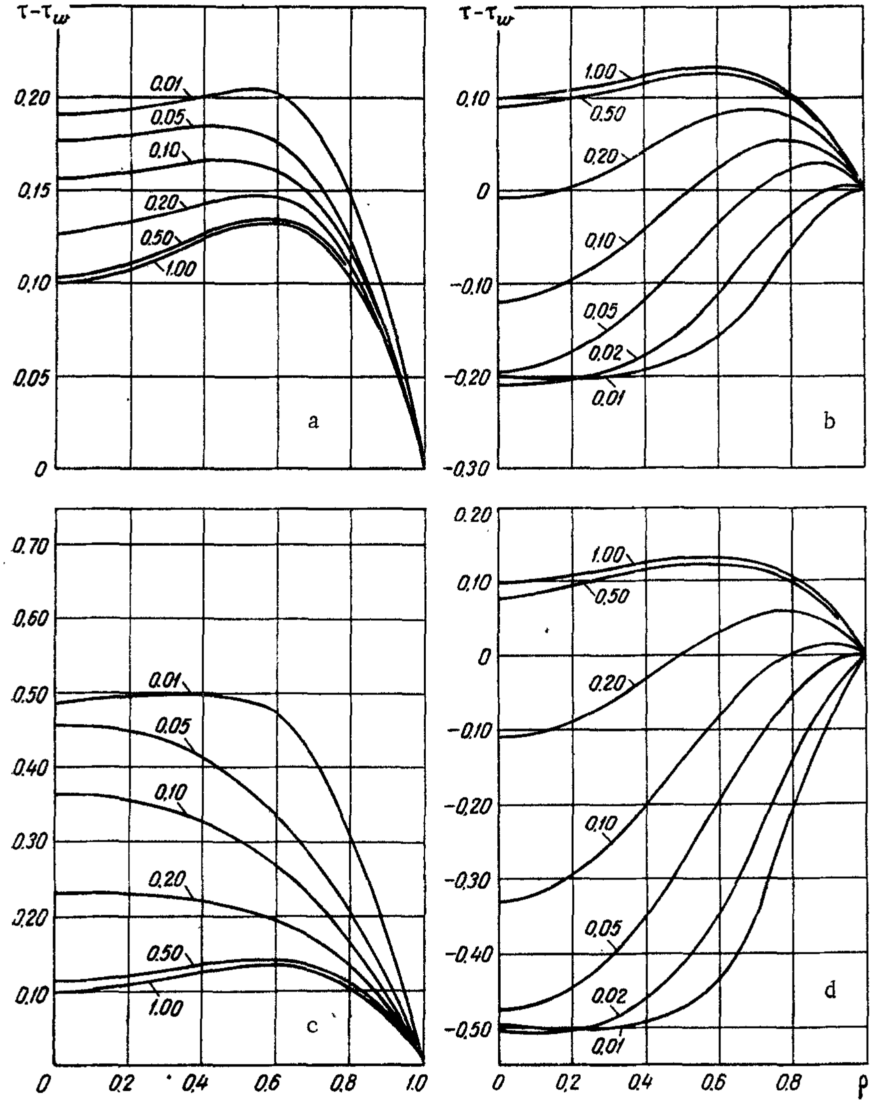

Fig. 2. Temperature distribution at $\varepsilon = 0.2$: a and b) at $\tau_0 - \tau_W = 0.2$ and -0.2; c and d) $\tau_0 - \tau_W = 0.5$ and -0.5. The figures on the curves are values of z.

we write the solution of (15) in the form

$$\theta_2 = -\sum_{i=1}^{\infty} A_i \exp(-\lambda_i z)\Phi_i(\rho), \tag{16}$$

where A_i are constants, λ_i eigenvalues, and Φ_i eigenfunctions of the problem which, as it is easy to show, satisfy the orthogonality condition

$$\int_0^1 (1-\rho^2)\rho\Phi_i\Phi_j\, d\rho = 0. \tag{17}$$

The general solution of Eq. (12) then assumes the form

$$\theta = -\sum_{i=1}^{\infty} A_i \exp(-\lambda_i z)\Phi_i(\rho) + \\ + \frac{1}{4}[(1+\varepsilon)(1-\rho^4) - 4\varepsilon(1-\rho^2)]\frac{1}{(\tau_0-\tau_w)}. \tag{18}$$

The eigenvalues λ_i and eigenfunctions Φ_i for the given problem are known, and the constants A_i are found on the basis of Eq. (18) for the boundary condition z = 0 starting from the orthogonality conditions

$$A_i = \frac{\int_0^1 (1-\rho^2)\rho\Phi_i\, d\rho}{\int_0^1 (1-\rho^2)\rho\Phi_i^2\, d\rho} + \frac{1}{4}\frac{\int_0^1 (\rho-\rho^3-\rho^5+\rho^7)\Phi_i\, d\rho}{\int_0^1 (1-\rho^2)\rho\Phi_i^2\, d\rho} \times \\ \times \frac{1}{(\tau_0-\tau_w)} + \frac{\varepsilon}{4}\frac{\int_0^1 (-3\rho+7\rho^3-5\rho^5+\rho^7)\Phi_i\, d\rho}{\int_0^1 (1-\rho^2)\rho\Phi_i^2\, d\rho} \times \\ \times \frac{1}{(\tau_0-\tau_w)}, \tag{19}$$

$$A_i = \left[-A_i' + A_i''\frac{1}{(\tau_0-\tau_w)} + \varepsilon A_i'''\frac{1}{(\tau_0-\tau_w)}\right]. \tag{20}$$

Returning to the original manner of expressing the relative temperature, we obtain the final solution

$$(\tau-\tau_w) = -\sum_{i=1}^{\infty}[A_i'(\tau_0-\tau_w) + A_i'' + \varepsilon A_i'''] \times \\ \times \exp(-\lambda_i z)\Phi_i(\rho) + \\ + \frac{1}{4}[(1+\varepsilon)(1-\rho^4) - 4\varepsilon(1-\rho^2)]. \tag{21}$$

When the wall temperature is equal to the temperature of the liquid at the inlet to the heat transfer section, Eq. (21) becomes

$$(\tau-\tau_0) = -\sum_{i=1}^{\infty}(A_i'' + \varepsilon A_i''')\exp(-\lambda_i z)\Phi_i(\rho) + \\ + \frac{1}{4}[(1+\varepsilon)(1-\rho^4) - 4\varepsilon(1-\rho)]. \tag{22}$$

Moreover, for an incompressible liquid we obtain

$$(\tau-\tau_0) = -\sum_{i=1}^{\infty} A_i'' \exp(-\lambda_i z)\Phi_i(\rho) + \frac{1}{4}(1-\rho^4). \tag{23}$$

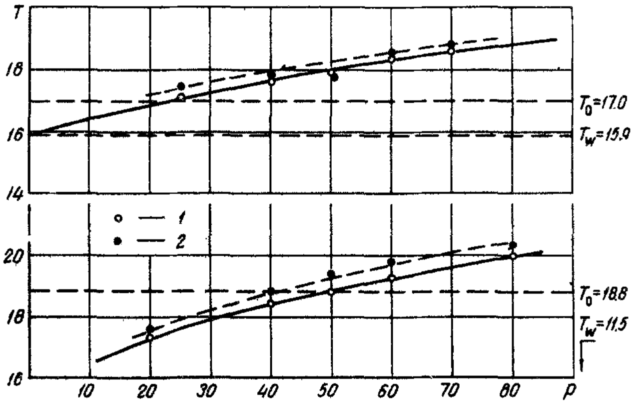

Fig. 3. Mean temperature of liquid: 1) calculated; 2) determined experimentally; T in °C, p in atm abs.

The last equation is identical to Brinkman's equation [7]. In the case of large values it leads to Hersey's simplified solution [2].

B. Wall temperature varies linearly. In this case the following boundary conditions hold:

$$\vartheta(\rho, z), \quad \vartheta(\rho, 0) = 0, \quad \left(\frac{\partial\vartheta}{\partial\rho}\right)_{\rho=0} = 0, \\ \vartheta(1, z) = \vartheta_w = az. \tag{24}$$

Now by the relative temperature ϑ we understand the temperature rise above the initial temperature of the liquid:

$$\vartheta = \tau - \tau_0.$$

Introducing into Eq. (8) the new variable

$$\Delta = \vartheta - \vartheta_w = \vartheta - az \tag{25}$$

and, moreover, setting $a = 4q$, we arrive at the equation

$$(1-\rho^2)\frac{\partial\Delta}{\partial\rho} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Delta}{\partial\rho}\right) + \\ + 4[(1+\varepsilon)\rho^2 - 2(1-\rho^2) - \varepsilon], \tag{26}$$

with the boundary conditions

$$\Delta(\rho, z), \quad \Delta(\rho, 0) = 0, \quad \left(\frac{\partial\Delta}{\partial\rho}\right)_{\rho=0} = 0; \\ \Delta(1, 0) = 0. \tag{27}$$

As before, the general solution may be represented as a sum of two functions:

$$\Delta = \Delta_1 + \Delta_2, \tag{28}$$

Δ_1 being the solution for $\partial\Delta/\partial z = 0$.

The general solution has the form

$$\Delta = -\sum_{i=1}^{\infty} A_i \exp(-\lambda_i z)\Phi_i(\rho) + \\ + \frac{1}{4}[(1+\varepsilon+2)(1-\rho^4) - 4(\varepsilon+2)(1-\rho^2)]. \tag{29}$$

The constants A_i are determined by the method previously employed:

$$A_i = \frac{1}{4}\frac{\int\limits_0^1 (\rho-\rho^3-\rho^5+\rho^7)\Phi_i d\rho}{\int\limits_0^1 (1-\rho^2)\rho\Phi_i^2 d\rho} + \\ + \frac{\varepsilon}{4}\frac{\int\limits_0^1 (-3\rho+7\rho^3-5\rho^5+\rho^7)\Phi_i d\rho}{\int\limits_0^1 (1-\rho^2)\rho\Phi_i^2 d\rho} + \\ + \frac{2}{4}\frac{\int\limits_0^1 (-3\rho+7\rho^3-5\rho^5+\rho^7)\Phi_i d\rho}{\int\limits_0^1 (1-\rho^2)\rho\Phi_i^2 d\rho}. \tag{30}$$

Setting

$$A_i = A_i'' + \varepsilon A_i''' + 2A_i''', \tag{31}$$

we arrive at the final form of the solution:

$$\vartheta = -\sum_{i=1}^{\infty}(A_i'' + \varepsilon A_i''' + 2A_i''')\exp(-\lambda_i z)\Phi_i(\rho) + \\ + \frac{1}{4}[(1+\varepsilon+2)(1-\rho^4) - 4(\varepsilon+2)(1-\rho^2)] + 42z. \tag{32}$$

The constants A_i'' and A_i''', the eigenvalues λ_i, and the eigenfunctions Φ_i remain as in the previous example.

For $q = 0$, i. e., for constant wall temperature equal to the temperature of the liquid at the inlet, Eq. (32) becomes identical with Eq. (22).

Calculation of temperature distribution. The constants λ_i and A_i in (21) are the characteristic quantities of the classical Grätz problem. The numerical values of these coefficients are well known and can be found in the literature. In the subsequent calculations we have used the eigenvalues λ_i and associated constants A_i determined by Abramovich (and also by Madejski [19]).

The calculations were made using a digital computer with allowance for the first four eigenfunctions. Values of A_i'' and A_i''' were determined from Eq. (19). The results of the calculations are presented in Table 1 (the first two columns give Abramovich's values).

Using the above values of λ_i and A_i, on the basis of Eqs. (21) and (22) it is possible to determine the temperature profile for a cross section of the channel represented by a certain transformed variable length z. Tables of values of Φ for the first four eigenfunctions, calculated for $\rho = 0.05$, are presented in [20].

The results of calculations for a capillary with constant wall temperature have been graphically represented in the form of the relationship between $(\tau - \tau_w)$ and ρ for various values of z. Figures 1 and 2 show the temperature distributions for several values of ε.* Obviously, the negative thermal effect of expansion reaches a maximum on the axis, and the positive effect of the heat of friction at the channel wall. The position of the temperature maxima relative to the channel axis depends on the parameter ε. Typical is the formation of an identical temperature distribution profile irrespective of the wall temperature, i. e., irrespective of whether it is kept at the inlet temperature, cooled, or heated.

CALCULATION OF MEAN TEMPERATURES

The mean relative temperature at a given section can be calculated from the equation

$$\vartheta_i = 4\int\limits_0^1 \vartheta(1-\rho^2)\rho\, d\rho = 4\left[\int\limits_0^1 \rho\vartheta\, d\rho - \int\limits_0^1 \rho^3\vartheta\, d\rho\right]. \tag{33}$$

Both the integrals in this equation were calculated and tabulated for capillaries with constant wall temperature.

Table 2 serves for calculating the mean temperature from Eq. (33). The following notation has been used:

$$M = \int\limits_0^1 \rho\vartheta\, d\rho; \qquad N = \int\limits_0^1 \rho^3\vartheta\, d\rho.$$

The values of M and N in the table are given as functions of the parameters ε, $(\tau_0 - \tau_w)$, z.

In using the table it is necessary to check the extent to which the length of the capillary exceeds the length of the acceleration (inlet) section:

$$\frac{x}{2R} \gg 0.029\,\mathrm{Re}.$$

On the basis of this relation and Eq. (6) we obtain the corresponding condition for the transformed length:

$$z \gg \frac{0.058}{\mathrm{Pr}}.$$

Since the symbol ϑ under the integral sign is understood as the difference $(\tau_0 - \tau_w)$ in accordance with Eq. (21), in order to determine the actual temperature of the liquid it is necessary to add to the calculated value τ_w for a capillary with a constant arbitrary wall temperature or τ_0 in the case $\tau_w = \tau_0$.

EXPERIMENTAL

The theoretical calculations of the mean temperature of the liquid (based on Eqs. (33) and (21), were checked experimentally for a capillary with constant wall temperature. The measurements were made in an apparatus consisting of a heating tank and a thermostated capillary [20].

In the experiments we used castor oil flowing through a brass capillary of diameter $d_0 = 0.216$ cm, wall thickness 0.04 cm, and length L = 36.6 cm cooled externally by a rapid isothermal flow of water.

In this case it may be assumed that the liquid flows through the capillary at constant wall temperature equal to τ_w. The oil was collected in a dewar flask, and the temperature was measured directly with a thermometer correct to 0.1° C. Small linear jet velocities

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*Curves for other values of ε and temperature drops $(\tau_0 - \tau_w)$ are presented in [20].

were selected, in order to eliminate corrections for the influence of kinetic energy in the measured thermal effect.

Under the experimental conditions the physical properties of the oil were as follows:

$\mu(p)$	ε	λ	γ	c_p
10.9—12.65	0.2	$1.81 \cdot 10^4$	0.962	$2.15 \cdot 10^7$

The remaining experimental parameters were varied within the following limits: expulsion pressure $(1.96-7.85) \cdot 10^7$ dyne/cm^2 (20–80 atm abs), relative length $z = 0.00473-0.0189$, relative difference $(\tau_0 - \tau_w) = 0.440-0.0061$.

The mean relative temperature was determined from Eq. (33) using calculated and tabulated integrals $\int_0^1 \rho\vartheta\, d\rho$ and $\int_0^1 \rho^3\vartheta\, d\rho$.

The results were graphically interpolated (correct to three places). Having determined the mean relative temperature ϑ_b for the required values of the parameters T_b and $(\tau_0 - \tau_w)$, we then calculate the mean temperature T_b.

The calculated curves of T_b as a function of pressure (i. e., essentially as a function of the shear stresses) and the points representing the results of the measurements are presented in Fig. 3.

The discrepancy between the theoretical and experimental curves can be explained if it is kept in mind that the postulate of constant wall temperature was realized only approximately (clearly, in practice the wall temperature was higher than the temperature of the cooling water). Determination of the mean temperature using the tables presented above makes it possible to develop a practical method suitable, for example, for estimating the magnitude of the thermal effects in capillary viscometry.

NOTATION

A_i are constants; c_p is the isobaric specific heat; D/Dt is the total derivative; Gz is the Grätz number; L is the length of capillary; M is the value of integral; N is the value of integral; p is the pressure; P is the pressure drop; q is the constant; R is the radius of capillary; r is the variable radius; T is the temperature, °C; t is the temperature rise relative to the reference temperature; t is the time, sec [sic]; T_b is the mean temperature; v is the specific volume; V is the velocity in axial direction, cm/sec; x stands for coordinates along channel axis, length, cm; z is the transformed (relative) length (Eq. (6)); β is the coefficient of thermal expansion; γ is the density; τ is the transformed (relative) temperature (Eq. (7)); τ_0 is the transformed initial temperature; τ_w is the transformed wall temperature; ε is the product T_β; ϑ is the transformed temperature (Eq. (10)); ϑ is the transformed temperature rise; ϑ_b is the mean transformed temperature; λ is the thermal conductivity; λ_i are eigenvalues; μ is the viscosity; ρ is the transformed radius; Δ is the transformed temperature (Eq. (25)); Φ are eigenfunctions, dissipation functions; ψ are eigenfunctions.

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